Static and dynamic phase transitions in multidimensional voting models on continua

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A voting model (or a generalization of the Glauber model at zero temperature) on a multidimensional lattice is defined as a system composed of a lattice, each site of which is either empty or occupied by a single particle. The reactions of the system are such that two adjacent sites, one empty, the other occupied, may evolve to a state where both of these sites are either empty or occupied. The continuum version of this model in a *D*-dimensional region with a boundary is studied, and two general behaviors of such systems are investigated, the stationary behavior of the system, and the dominant way of relaxation of the system toward its stationary state. Based on the first behavior, a static phase transition (discontinuous changes in the stationary profiles of the system) is studied. Based on the second behavior, a dynamical phase transition (discontinuous changes in the relaxation times of the system) is studied. It is shown that the static phase transition is induced by the bulk reactions only, while the dynamical phase transition is a result of both bulk reactions and boundary conditions.

I. INTRODUCTION

The study of reaction-diffusion systems is an attractive area. A reaction-diffusion system consists of a collection of particles (of one or several species) moving and interacting with each other with specific probabilities (or rates in the case of a continuous time variable). In so called exclusion processes, any site of the lattice the particles move on is either vacant or occupied by one particle. The aim of studying such systems is of course to calculate their time evolution. But to find the complete time evolution of a reaction-diffusion system is generally a very difficult (if not impossible) task.

Various methods have been used to study the reaction-diffusion system: analytical techniques, approximation methods, and simulation. The success of the approximation methods may be different in different dimensions, as for example the mean field techniques, working well for high dimensions, generally do not give correct results for low-dimensional systems. A large fraction of analytical studies belong to low-dimensional (specially one-dimensional) systems, as solving low-dimensional systems should in principle be easier [1–12].

Various classes of reaction-diffusion systems are called exactly solvable, in different senses. In [13–15], integrability means that the *N*-particle conditional probability *S* matrix is factorized into a product of two-particle *S* matrices. This is related to the fact that for systems solvable in this sense, there are a large number of conserved quantities. In [16–24], solvability means closedness of the evolution equation of the empty intervals (or their generalization).

Consider a reaction-diffusion system (on a lattice) with open boundaries. By open boundaries, it is meant that in addition to the reactions in the bulk of the lattice, particles at the boundaries do react with some external source. A question is to find the possible phase transitions of the system. By phase transition, it is meant a discontinuity in some behavior of the system with respect to its parameters. Such discontinuities may arise in two general categories: in the stationary (large time) profiles of the system, and in the time constants determining the evolution of the system. In the first case, static phase transitions are dealt with; in the second case, dynamical phase transitions. For a review on dynamical phase transitions, one can see, for example, [25].

There are systems for which the equation of motion of the one-point function (the probability that a certain site be occupied) is closed, that is, independent of the more-point functions [26–28]. Among these systems is the so called voting model (or a generalization of the Glauber model at zero temperature). In [29] a voting system on a one-dimensional lattice was studied, for which at the boundaries of the lattice there is injection or extraction of the particles. Based on the evolution of the one-point functions, it was shown there that the system exhibits two kinds of phase transitions: a static phase transition, corresponding to a discontinuous change in the stationary profile of the one-point function; and a dynamical one, corresponding to a discontinuous change in the behavior of the relaxation time of the system toward its stationary state. In [30–34], the phase structures of extensions of such systems on a one-dimensional lattice were investigated. All of these are restricted to the case of a one-dimensional lattice.

Extending these investigations to higher-dimensional cases would be interesting. Here, we want to study a multidimensional extension of the voting model, on a continuum rather than a lattice.

The scheme of the present article is as follows. In Sec. II, the multidimensional voting model on a continuum is presented, and the evolution equation for the density of the particles is obtained. In Sec. III, the time independence of the system is studied and it is shown that the system exhibits a static phase transition. In Sec. IV, the relaxation of the sys-
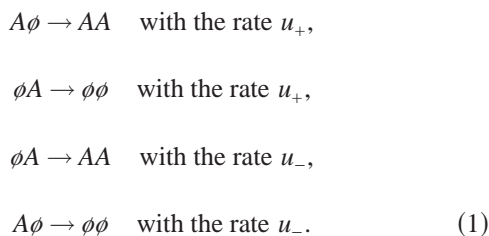
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tem toward its stationary state is studied and it is shown that the system exhibits a dynamical phase transition. Section V is devoted to the concluding remarks.

II. MULTIDIMENSIONAL VOTING MODELS ON CONTINUA

In [11,29], a one-dimensional voting model (or a generalization of the Glauber model at zero temperature) on a lattice was defined as follows. Let the system consist of a one-dimensional lattice, each of the sites of which is either empty (ϕ) or containing a single particle (A), and let there be a reaction between two neighboring sites such as

$$A\phi \rightarrow AA \quad \text{with the rate } u_+,$$

$$\phi A \rightarrow \phi\phi \quad \text{with the rate } u_+,$$

$$\phi A \rightarrow AA \quad \text{with the rate } u_-,$$

$$A\phi \rightarrow \phi\phi \quad \text{with the rate } u_-. \tag{1}$$

In [11], an open lattice was investigated while in [29] a lattice was studied at the boundaries of which injection and extraction of particles could occur. It was shown that these models are autonomous, meaning that the evolution equation of the n-point functions contain only n- or fewer-point functions. In fact, as was seen in [29],

$$\frac{d}{dt}\langle n_i \rangle = u_+ \langle n_{i-1} \rangle + u_- \langle n_{i+1} \rangle - (u_+ + u_-)\langle n_i \rangle. \tag{2}$$

Here n_i is the particle number operator at the site i of the lattice.

Now consider a multidimensional lattice, each site of which is either empty or occupied by a single particle, and let there be a reaction such as

$$A\phi \rightarrow AA \quad \text{with the rate } u_l,$$

$$\phi A \rightarrow \phi\phi \quad \text{with the rate } u_l. \tag{3}$$
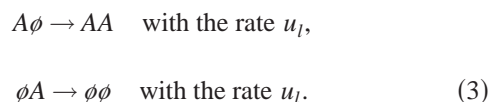
Here, we are considering the reaction between a site i (the right site), which is the ending point of the link l, and another site (the left site) which is the starting point of the same link. In a one-dimensional lattice, each site is the ending point of two links, which have been denoted by $+$ and $-$. From Eq. (3), it is seen that the evolution equation for the one-point function is

$$\frac{d}{dt}\langle n_i \rangle = \sum_l [u_l(\langle n_{i-l} \rangle - \langle n_{i-l}n_i \rangle) - u_l(\langle n_i \rangle - \langle n_{i-l}n_i \rangle)], \tag{4}$$

where by the site index $i-l$, is meant a site which is the starting point of the link l, the ending point of which is the site i. It is seen that the two-point functions in the right-hand side of Eq. (4) cancel each other. So

$$\frac{d}{dt}\langle n_i \rangle = \sum_l u_l(\langle n_{i-l} \rangle - \langle n_i \rangle). \tag{5}$$

Now, assume that the one-point function is a slowly varying function of its argument (i). In this case, one can define a smooth particle density function of the continuous position variable **r**, with

$$\rho(\mathbf{r}_i) := \frac{1}{\mathcal{V}}\langle n_i \rangle, \tag{6}$$

where $\mathbf{r}_i$ is the position of the lattice site i, and $\mathcal{V}$ is the *specific hypervolume* of the site. Then Eq. (5) can be rewritten as

$$\frac{\partial}{\partial t}\rho = \sum_l u_l \left[-\boldsymbol{\delta}_l \cdot \boldsymbol{\nabla} + \frac{1}{2}(\boldsymbol{\delta}_l \cdot \boldsymbol{\nabla})^2 \right]\rho, \tag{7}$$

where $\boldsymbol{\delta}_l$ is the link vector, equal to the position of the ending point of the link l minus the position of the starting point of the link l, and higher-derivative terms have been neglected. Using suitable coordinates for **r**, one can write the second derivative as

$$\frac{1}{2}\sum_l u_l(\boldsymbol{\delta}_l \cdot \boldsymbol{\nabla})^2 = \sum_a \left(\frac{\partial}{\partial x^a} \right)^2, \tag{8}$$

where the x^a's are the coordinates of **r**. So Eq. (7) is rewritten as

$$\frac{\partial}{\partial t}\rho = (-\mathbf{v} \cdot \boldsymbol{\nabla} + \nabla^2)\rho, \tag{9}$$

where

$$\mathbf{v} := \sum_l u_l \boldsymbol{\delta}_l. \tag{10}$$

Equation (9) is nothing but a diffusion equation combined with a drift velocity **v**.

Suppose that Eq. (9) holds for the interior of the region V. Integrating Eq. (9) on V, one arrives at

$$\frac{d}{dt}\int_V dV \rho = -\oint_{\partial V} dS\, \mathbf{n} \cdot \mathbf{v}\rho + \oint_{\partial V} dS\, \mathbf{n} \cdot \boldsymbol{\nabla}\rho. \tag{11}$$

The first term in the right-hand side is the rate of change of the total number of particles inside, as a consequence of the drift, while the second term is the effect of injecting or extracting particles at the boundary. The boundary condition

$$\mathbf{n} \cdot \boldsymbol{\nabla}\rho = \alpha - \beta\rho \quad \text{at the boundary} \tag{12}$$

corresponds to an injection rate of α per unit hyperarea of the boundary, and an extraction rate of β per unit hyperarea per particle density at the boundary. In general, one can take α and β position dependent.

Comparing Eqs. (9) and (12) with Eq. (7) of [29], it is seen that one can transform Eq. (7) of [29] to Eqs. (9) and (12) through

$$\delta = \sqrt{\frac{2}{u+v}},$$

$$\mathbf{v} \cdot \hat{\mathbf{x}} = \delta(u-v),$$

$$\alpha^- = \frac{a}{\delta u v},$$

$$\beta^- = \frac{a + a'}{\delta u},$$

$$\alpha^+ = \frac{b}{\delta u v},$$

$$\beta^+ = \frac{b + b'}{\delta v}, \tag{13}$$

where the right-hand sides are the quantities defined in [29], and the superscripts − and + refer to the left and right boundaries, respectively.

From now on, for simplicity we restrict ourselves to the case that the volume V is a D-dimensional hyperball with radius R, the boundary of which is a hypersphere.

III. THE TIME-INDEPENDENT STATE AND THE STATIC PHASE TRANSITION

Let ρ_0 be the time-independent solution to Eqs. (9) and (12). Using the ansatz

$$F_{\mathbf{q}}(\mathbf{r}) := \exp(\mathbf{q} \cdot \mathbf{r}) \tag{14}$$

(with $\mathbf{q}$ a constant vector) as a time-independent solution to Eq. (9), one arrives at

$$\mathbf{q} \cdot \mathbf{q} - \mathbf{v} \cdot \mathbf{q} = 0, \tag{15}$$

which leads to

$$\mathbf{q} = \frac{1}{2}(\mathbf{v} + \mathbf{v}'), \tag{16}$$

where $\mathbf{v}'$ is an arbitrary constant vector subject to the condition

$$\mathbf{v}' \cdot \mathbf{v}' = \mathbf{v} \cdot \mathbf{v}. \tag{17}$$

So one can write the general time-independent solution to Eq. (9) as

$$\begin{aligned}
\rho_0(\mathbf{r}) &= \int d\Omega' \tilde{A}(\Omega') F_{\mathbf{q}}(\mathbf{r}) \\
&= \int d\Omega' A(\Omega') \exp\left\{ \frac{1}{2}[(\mathbf{v} + \mathbf{v}') \cdot \mathbf{r} - |\mathbf{v} + \mathbf{v}'|R] \right\} \\
&=: \int d\Omega' A(\Omega') \exp[G(\mathbf{v}', \mathbf{r})],
\end{aligned} \tag{18}$$

where Ω' denotes the angular coordinates of $\mathbf{v}'$, and A is an arbitrary function. It is easy to see that the maximum value of G is zero, and this maximum value is reached at a point on the boundary ($r=R$), where $\mathbf{r}$ is parallel with $\mathbf{v}+\mathbf{v}'$.

For large values of R and r, G is a rapidly varying function and the integral is mainly determined from that point of the integration region which maximizes G. Generally, there may be two such points. One point is

$$\mathbf{v}'_1 = -\mathbf{v}. \tag{19}$$

The other point is

$$(\mathbf{v}'_2 + \mathbf{v}) \cdot \mathbf{r} = |\mathbf{v}'_2 + \mathbf{v}|r \quad \text{for } r = R, \tag{20}$$

which means that $\mathbf{q}$ is parallel with $\mathbf{r}$. As the angle between $\mathbf{q}$ and $\mathbf{v}$ cannot exceed $\pi/2$, the second point exists only if the angle between $\mathbf{r}$ and $\mathbf{v}$ is less than $\pi/2$. One has

$$G(\mathbf{v}'_1, \mathbf{r}) = 0 \tag{21}$$

and

$$\begin{aligned}
G[\mathbf{v}'_2(\mathbf{r}), \mathbf{r}] &= G[\mathbf{v}'_2(R \ \mathbf{r}/r), \mathbf{r}] + O[(R - r)^2] \\
&= \frac{(r - R)|\mathbf{v} + \mathbf{v}'_2|}{2} + O[(r - R)^2] \\
&= \frac{(r - R)\mathbf{v} \cdot \mathbf{r}}{R} + O[(r - R)^2].
\end{aligned} \tag{22}$$

Using Eqs. (21) and (22), one arrives at

$\rho_0(\mathbf{r})$

$$\sim \begin{cases} C_1(\Omega), & r \sim R, \ \mathbf{r} \cdot \mathbf{v} < 0, \\ C_1(\Omega) + C_2(\Omega)\exp\left[\dfrac{(r-R)\mathbf{v} \cdot \mathbf{r}}{R}\right], & r \sim R, \ \mathbf{r} \cdot \mathbf{v} > 0. \end{cases} \tag{23}$$

From this,

$$\boldsymbol{\nabla}\rho_0(r = R) \propto \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{v})\theta(\boldsymbol{n} \cdot \boldsymbol{v}), \quad R \to \infty, \tag{24}$$

where θ is the step function. It is seen that in the thermodynamic limit ($R \to \infty$), the density profile at the boundary is stationary, unless $\boldsymbol{v} \cdot \boldsymbol{r} > 0$. So, changing $\boldsymbol{v}$ one can induce a discontinuous change in the slope of the profile density at the boundary. This is the static phase transition, which is seen to be independent of the injection and extraction terms, but dependent on the drift velocity.

IV. THE RELAXATION OF THE SYSTEM TOWARD THE STATIONARY STATE, AND THE DYNAMIC PHASE TRANSITION

Starting from Eqs. (9) and (12), one arrives at

$$\frac{\partial}{\partial t}(\rho - \rho_0) = (-\mathbf{v} \cdot \boldsymbol{\nabla} + \nabla^2)(\rho - \rho_0) =: h(\rho - \rho_0) \tag{25}$$

and

$$\mathbf{n} \cdot \boldsymbol{\nabla}(\rho - \rho_0) = -\beta(\rho - \rho_0) \quad \text{at the boundary} \tag{26}$$

where ρ_0 is the time-independent solution to Eqs. (9) and (12). Let ψ be an eigenfunction of h corresponding to the eigenvalue E. Using the ansatz (14) in the eigenvalue equation corresponding to h, one arrives at

$$\mathbf{q} \cdot \mathbf{q} - \mathbf{v} \cdot \mathbf{q} = E, \tag{27}$$

which leads to

$$\mathbf{q} = \frac{1}{2}(\mathbf{v} + \mathbf{v}'), \qquad (28)$$

where $\mathbf{v}'$ is an arbitrary constant vector subject to the condition

$$\mathbf{v}' \cdot \mathbf{v}' = \mathbf{v} \cdot \mathbf{v} + 4E. \qquad (29)$$

So one has

$$\psi(\mathbf{r}) = \exp(\mathbf{v} \cdot \mathbf{r}/2) \int d\Omega' A(\Omega') \exp(\mathbf{v}' \cdot \mathbf{r}/2), \qquad (30)$$

where A is to be found so that the boundary condition (26) is satisfied with ψ.

If the right-hand side of Eq. (29) is positive, then $\mathbf{v}'$ is real, and for large r, one can approximate ψ as

$$\psi(\mathbf{r}) \sim \exp(\mathbf{v} \cdot \mathbf{r}/2) A(\Omega) \exp(v' r/2), \qquad (31)$$

where Ω are the angular coordinates corresponding to $\mathbf{r}$. The boundary condition (16) then becomes

$$\left[\frac{v'}{2} + \beta(\Omega) + \frac{\mathbf{n} \cdot \mathbf{v}}{2} \right] A(\Omega) = 0. \qquad (32)$$

This has a nonzero solution for A, provided the expression in brackets vanishes for some Ω. As $\beta \geq 0$, this happens for some (real) positive v', if and only if

$$\min\left[\beta(\Omega) + \frac{v \cos\phi}{2} \right] < 0, \qquad (33)$$

where ϕ is the angle between $\mathbf{r}$ and $\mathbf{v}$. If (33) holds, then the range of v' for which a nonzero solution to Eq. (32) for A exists is

$$0 \leq v' \leq -\min\left[\beta(\Omega) + \frac{v \cos\phi}{2} \right]. \qquad (34)$$

[This is true for more than one dimension. If the space is one dimensional, v' has only one acceptable value, as the expression in brackets in Eq. (32) has only two values, at most one of which can be zero.]

If (33) holds, then there exist eigenvalues E for h, with $E > -\mathbf{v} \cdot \mathbf{v}/4$. Otherwise, all of the eigenvalues of h are less than or equal to $-\mathbf{v} \cdot \mathbf{v}/4$. The relaxation time of the system is

$$\tau = -\frac{1}{E_{\max}}, \qquad (35)$$

where $E_{\max}$ is the largest eigenvalue of h. The largest value of E is either $-\mathbf{v} \cdot \mathbf{v}/4$, or the value obtained from Eq. (29) for the largest value of v'. So

$$\tau = \begin{cases} \dfrac{4}{\mathbf{v} \cdot \mathbf{v}}, & \min\left[\beta(\Omega) + \dfrac{v \cos\phi}{2} \right] > 0, \\[4mm] \dfrac{4}{\mathbf{v} \cdot \mathbf{v} - \{\min[2\beta(\Omega) + v \cos\phi]\}^2}, & \min\left[\beta(\Omega) + \dfrac{v \cos\phi}{2} \right] < 0. \end{cases} \qquad (36)$$

In the first case, the system is in the fast dynamical phase, in which the relaxation time does not depend on the boundary conditions. In the second case, the system is in the slow dynamical phase, in which the relaxation time is larger and does depend on the boundary conditions. This is the dynamical phase transition.

V. CONCLUDING REMARKS

It was seen that the so-called voting model defined on a one-dimensional lattice has a natural analog on a multidimensional continuum. It was seen that there are two kinds of phase transition, a static one corresponding to a discontinuous change in the behavior of the stationary profile of the system, and a dynamical phase transition corresponding to the relaxation of the system toward its stationary state. The static phase transition is controlled by the bulk reactions, while the dynamical phase transition is controlled by the bulk reactions and the boundary conditions both. This is analogous to what is seen for the case of a one-dimensional lattice.

There are, however, differences. In the multidimensional case, the static transition occurs when the direction of the drift velocity is changed. This can happen without it being necessary that the drift velocity vanishes. In the one-dimensional case, however, the static phase transition occurs only when the drift velocity passes zero. The reason is that in one dimension the only way to change the direction of a vector smoothly is that the vector vanishes at some point.

The second difference concerns the dynamical phase transition; to be more precise, the largest eigenvalues of the operator h defined in Eq. (25). In the one-dimensional case and in the slow phase, there is only one eigenvalue greater than the largest eigenvalue corresponding to the fast phase. In the multidimensional case, however, in the slow phase the spectrum of h contains a continuous region the lower bound of which is the largest eigenvalue of h in the fast phase. This means that in the one-dimensional case and in the slow phase, there is a largest relaxation time and a gap between this and the next largest relaxation time, while in the multidimensional case, there is no such gap.

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